

**What Is Claimed Is:**

1           1.       A method for using interval techniques within a computer system  
2       to solve a multi-objective optimization problem, comprising:  
3           receiving a representation of multiple objective functions ( $f_1, \dots, f_n$ ) at  
4       the computer system, wherein ( $f_1, \dots, f_n$ ) are scalar functions of a vector  
5        $\mathbf{x} = (x_1, \dots, x_n)$ ;  
6           receiving a representation of a domain of interest for the multiple objective  
7       functions;  
8           storing the representations in a memory within the computer system; and  
9           performing an interval optimization process to compute guaranteed bounds  
10       on a Pareto front for the objective functions ( $f_1, \dots, f_n$ ), wherein for each point  
11       on the Pareto front, an improvement in one objective function cannot be made  
12       without adversely affecting at least one other objective function;  
13           wherein performing the interval optimization process involves applying a  
14       direct-comparison technique between subdomains of the domain of interest to  
15       eliminate subdomains that are certainly dominated by other subdomains.

1           2.       The method of claim 1, wherein performing the interval  
2       optimization process involves applying a gradient technique to eliminate  
3       subdomains that do not contain a local Pareto optimum.

1           3.       The method of claim 2, wherein a subdomain  $[\mathbf{x}]_i$  is eliminated by  
2       the gradient technique if an intersection of certainly negative gradient regions  $\mathbf{C}_j$   
3       for each objective function  $f_j$  is non-empty,  $\bigcap_{j=1}^n \mathbf{C}_j([\mathbf{x}]_i) \neq \emptyset$ ;

4            wherein the certainly negative gradient region  $C_j$  for objective function  $f_j$  is  
5   the intersection of  $\underline{N}_j([x]_i)$  (the negative gradient region associated with the  
6   minimum angle  $\underline{\theta}_j$  of the gradient of  $f_j$  over the subdomain  $[x]_i$ ) and  $\overline{N}_j([x]_i)$  (the  
7   negative gradient region associated with the maximum angle  $\overline{\theta}_j$  of the gradient of  
8    $f_j$  over the subdomain  $[x]_i$ ).

1            4.        The method of claim 3, wherein the method further comprises  
2   iteratively:  
3            bisecting remaining subdomains that have not been eliminated by the  
4   gradient technique; and  
5            applying the gradient technique to eliminate bisected subdomains that do  
6   not contain a local Pareto optimum.

1            5.        The method of claim 4, wherein bisecting a subdomain involves  
2   bisecting the subdomain in the direction that has the largest width of partial  
3   derivatives of all objective functions  $(f_1, \dots, f_n)$  over the subdomain.

1            6.        The method of claim 4, wherein the direct-comparison technique is  
2   applied once for every  $n$  iterations of the gradient technique.

1            7.        The method of claim 6, wherein the iterations continue until either  
2   a predetermined maximum number of iterations are performed, or the largest area  
3   of any subdomain is below a predetermined value.

1            8.        The method of claim 1,

2 wherein a subdomain **U** certainly dominates a subdomain **V** if every point  
3 **u**  $\in$  **U** dominates every point **v**  $\in$  **V**; and  
4 wherein a point **u** dominates a point **v** under minimization if,  
5  $u_i \leq v_i, i = 1, \dots, n$ , and  
6  $u_i < v_i$  for some  $i \in \{1, \dots, n\}$ .

1 9. A computer-readable storage medium storing instructions that  
2 when executed by a computer cause the computer to perform a method for using  
3 interval techniques within a computer system to solve a multi-objective  
4 optimization problem, the method comprising:  
5 receiving a representation of multiple objective functions ( $f_1, \dots, f_n$ ) at  
6 the computer system, wherein ( $f_1, \dots, f_n$ ) are scalar functions of a vector  
7  $\mathbf{x} = (x_1, \dots, x_n)$ ;  
8 receiving a representation of a domain of interest for the multiple objective  
9 functions;  
10 storing the representations in a memory within the computer system; and  
11 performing an interval optimization process to compute guaranteed bounds  
12 on a Pareto front for the objective functions ( $f_1, \dots, f_n$ ), wherein for each point  
13 on the Pareto front, an improvement in one objective function cannot be made  
14 without adversely affecting at least one other objective function;  
15 wherein performing the interval optimization process involves applying a  
16 direct-comparison technique between subdomains of the domain of interest to  
17 eliminate subdomains that are certainly dominated by other subdomains.

1 10. The computer-readable storage medium of claim 9, wherein  
2 performing the interval optimization process involves applying a gradient  
3 technique to eliminate subdomains that do not contain a local Pareto optimum.

1            11.    The computer-readable storage medium of claim 10, wherein a  
2    subdomain  $[\mathbf{x}]_i$  is eliminated by the gradient technique if an intersection of  
3    certainly negative gradient regions  $\mathbf{C}_j$  for each objective function  $f_j$  is non-empty,  
4     $\bigcap_{j=1}^n \mathbf{C}_j([\mathbf{x}]_i) \neq \emptyset$ ;  
5            wherein the certainly negative gradient region  $\mathbf{C}_j$  for objective function  $f_j$  is  
6    the intersection of  $\underline{\mathbf{N}}_j([\mathbf{x}]_i)$  (the negative gradient region associated with the  
7    minimum angle  $\underline{\theta}_j$  of the gradient of  $f_j$  over the subdomain  $[\mathbf{x}]_i$ ) and  $\overline{\mathbf{N}}_j([\mathbf{x}]_i)$  (the  
8    negative gradient region associated with the maximum angle  $\overline{\theta}_j$  of the gradient of  
9     $f_j$  over the subdomain  $[\mathbf{x}]_i$ ).

1            12.    The computer-readable storage medium of claim 11, wherein the  
2    method further comprises iteratively:  
3            bisecting remaining subdomains that have not been eliminated by the  
4    gradient technique; and  
5            applying the gradient technique to eliminate bisected subdomains that do  
6    not contain a local Pareto optimum.

1            13.    The computer-readable storage medium of claim 12, wherein  
2    bisecting a subdomain involves bisecting the subdomain in the direction that has  
3    the largest width of partial derivatives of all objective functions  $(f_1, \dots, f_n)$  over  
4    the subdomain.

1           14.     The computer-readable storage medium of claim 12, wherein the  
2     direct-comparison technique is applied once for every  $n$  iterations of the gradient  
3     technique.

1           15.     The computer-readable storage medium of claim 14, wherein the  
2     iterations continue until either a predetermined maximum number of iterations are  
3     performed, or the largest area of any subdomain is below a predetermined value.

1           16.     The computer-readable storage medium of claim 9,  
2             wherein a subdomain  $U$  certainly dominates a subdomain  $V$  if every point  
3      $u \in U$  dominates every point  $v \in V$ ; and  
4             wherein a point  $u$  dominates a point  $v$  under minimization if,  
5                      $u_i \leq v_i, i = 1, \dots, n$ , and  
6                      $u_i < v_i$  for some  $i \in \{1, \dots, n\}$ .

1           17.     An apparatus that uses interval techniques to solve a multi-  
2     objective optimization problem, comprising:  
3             a receiving mechanism configured to receive a representation of multiple  
4     objective functions  $(f_1, \dots, f_n)$ , wherein  $(f_1, \dots, f_n)$  are scalar functions of a  
5     vector  $\mathbf{x} = (x_1, \dots, x_n)$ ;  
6             wherein the receiving mechanism is configured to receive a representation  
7     of a domain of interest for the multiple objective functions;  
8             a memory configured to store the representations; and  
9             an interval optimizer configured to performing an interval optimization  
10    process to compute guaranteed bounds on a Pareto front for the objective  
11    functions  $(f_1, \dots, f_n)$ , wherein for each point on the Pareto front, an

improvement in one objective function cannot be made without adversely affecting at least one other objective function; wherein the interval optimizer is configured to apply a direct-comparison technique between subdomains of the domain of interest to eliminate subdomains that are certainly dominated by other subdomains.

18. The apparatus of claim 17, wherein the interval optimizer is configured to apply a gradient technique to eliminate subdomains that do not contain a local Pareto optimum.

19. The apparatus of claim 18, wherein a subdomain  $[\mathbf{x}]_i$  is eliminated by the gradient technique if an intersection of certainly negative gradient regions  $\mathbf{C}_j$  for each objective function  $f_j$  is non-empty,  $\bigcap_{j=1}^n \mathbf{C}_j([\mathbf{x}]_i) \neq \emptyset$ ; wherein the certainly negative gradient region  $\mathbf{C}_j$  for objective function  $f_j$  is the intersection of  $\mathbf{N}_j([\mathbf{x}]_i)$  (the negative gradient region associated with the minimum angle  $\underline{\theta}_j$  of the gradient of  $f_j$  over the subdomain  $[\mathbf{x}]_i$ ) and  $\overline{\mathbf{N}}_j([\mathbf{x}]_i)$  (the negative gradient region associated with the maximum angle  $\overline{\theta}_j$  of the gradient of  $f_j$  over the subdomain  $[\mathbf{x}]_i$ ).

20. The apparatus of claim 19, wherein the interval optimizer is configured to iteratively: bisect remaining subdomains that have not been eliminated by the gradient technique; and to apply the gradient technique to eliminate bisected subdomains that do not contain a local Pareto optimum.

1           21.     The apparatus of claim 20, wherein bisecting a subdomain involves  
2     bisecting the subdomain in the direction that has the largest width of partial  
3     derivatives of all objective functions ( $f_1$ , ...,  $f_n$ ) over the subdomain.

1           22.     The apparatus of claim 20, wherein the direct-comparison  
2     technique is applied once for every  $n$  iterations of the gradient technique.

1           23.     The apparatus of claim 22, wherein the iterations continue until  
2     either a predetermined maximum number of iterations are performed, or the  
3     largest area of any subdomain is below a predetermined value.

1           24.     The apparatus of claim 17,  
2             wherein a subdomain  $U$  certainly dominates a subdomain  $V$  if every point  
3      $u \in U$  dominates every point  $v \in V$ ; and  
4             wherein a point  $u$  dominates a point  $v$  under minimization if,  
5                      $u_i \leq v_i, i = 1, \dots, n$ , and  
6                      $u_i < v_i$  for some  $i \in \{1, \dots, n\}$ .